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Abstract

We study a generic minimization problem with separable non-convex piecewise linear costs, showing that the linear programming (LP) relaxation of three textbook mixed integer programming formulations each approximates the cost function by its lower convex envelope. We also show their equivalence by translating solutions between the three LP relaxations.

Key words : piecewise-linear, integer programming, linear relaxation

Résumé

Nous considérons un problème de minimisation générique dans lequel l'objectif consiste d'une somme séparable de fonctions linéaires par morceaux non convexes. Nous montrons que les relaxations linéaires de trois modèles classiques de programmation en nombres entiers sont équivalentes puisqu'elles fournissent comme approximation de l'objectif son enveloppe convexe inférieure. Nous établissons également cette équivalence en montrant comment obtenir une solution de l'une des relaxations à partir d'une solution des autres.

Mots-clés : linéaire par morceaux, progrmmation en nombres entiers, relaxation linéaire

1 Introduction

Optimization problems with piecewise linear costs arise in many application domains, including transportation, telecommunications, and production planning. In particular, many researchers have studied the minimum cost network flow problem with non-convex piecewise linear costs [1] [2][5][6][7][9]. Specific applications include the network loading problem [3][4][12][13][17], the facility location problem with multiple capacity options [15][16], and the merge-in-transit problem [8]. Each of these studies introduces integer variables to model the costs, though choice of the basic formulation varies and includes three textbook models, the so-called incremental, multiple choice and convex combination models. The objective of this paper is to show that the linear programming (LP) relaxations of these mixed-integer programming (MIP) models are equivalent in the sense that they all approximate the cost function by its lower convex envelope. To the best of our knowledge, although this result might appear to be intuitive, no one has formally established it.

We consider an arbitrary piecewise linear cost function like the one in Figure 1. The cost, $g(x)$, is a function of a single variable, x, with the variable and fixed costs varying according to the value of x. The function need not be continuous; it can have positive or negative jumps, though we do assume that the function is lower semi-continuous, that is, $g(x) \leq \liminf_{x' \to x} g(x')$. Without loss of generality, we also assume, through a simple translation of the costs if necessary, that $g(0) = 0$.

The general problem is to minimize the separable sum of n piecewise linear functions, subject to linear constraints, which we write as,

$$\text{Minimize}_{\mathbf{x} \in L}\ g(\mathbf{x}) \tag{1}$$

whose objective function, $g(\mathbf{x}) = \sum_{j=1}^{n} g_j(x_j)$, is defined over a bounded polyhedron, L, in $\Re^n_+ = \{\mathbf{x} \in \Re^n | \mathbf{x} \geq 0\}$. Since the formulations we consider model each function $g_j(x_j)$ separately, for notational simplicity, we will drop the subscript j. We then let x denote a single variable and focus on a single piecewise linear function $g(x)$. We will refer to the value of x as the *load* on the variable.

Each piecewise linear segment s of the function $g(x)$ has a variable cost, c^s (the slope), a fixed cost, f^s (the cost-intercept), and upper and lower bounds, b^{s-1} and b^s (the breakpoints), on the load corresponding to that segment. We assume $b^0 = 0$. Figure 2 illustrates the notation.

Using this notation, in Section 2 we present three well-known valid MIP models for the problem. In Section 3, we show that the LP relaxation of each formulation approximates the actual cost function by its lower convex envelope, and thus the LP relaxations of the three formulations are equivalent. We further establish this equivalency by providing a translation between the LP solutions from one formulation to each of the others.

2 Three Models for Piecewise Linear Costs

To set notation, we introduce a formulation for each of the three models that we examine.

2.1 Incremental Model

Dantzig [10] and Vajda [19] both attribute the incremental model to a 1957 paper by Manne and Markowitz [18]. As reported in early textbooks, including those by Dantzig [11] in 1963 and Hadley [14] in 1964, the incremental model introduces a segment load variable, x^s, for each segment, defined as the load on the segment s. The final value of the variable, or load, $x = \sum_s x^s$, is the sum of the incremental quantities in each segment.

To be feasible using this variable definition, the value on segment $s+1$ must be zero unless segment s is "full," that is, $x^{s+1} > 0$ only if $x^s = b^s - b^{s-1}$. To account for this requirement, the incremental model introduces binary variables, y^s, defined by the condition that $y^s = 1$ if $x^s > 0$, and $y^s = 0$ otherwise. Defining $\widehat{f}^s = (f^s + c^s b^{s-1}) - (f^{s-1} + c^{s-1} b^{s-1})$ as the gap in the cost at the breakpoint between segment $s-1$ and segment s, we can now express the problem given by (1) as a MIP formulation by writing the objective function as $g(x) = \sum_s c^s x^s + \widehat{f}^s y^s$, with the additional constraints:

$$x = \sum_s x^s \tag{2}$$

$$(b^s - b^{s-1})y^{s+1} \leq x^s \leq (b^s - b^{s-1})y^s \tag{3}$$

$$y^s \in \{0,1\}. \tag{4}$$

In the remainder of the paper, we refer to the system defined by (2)-(4) as I_x and its LP relaxation as $LP(I_x)$.

2.2 Multiple Choice Model

The multiple choice model, as used by Balakrishnan and Graves [2], among others, employs an alternative definition of the segment variables. In this formulation, x^s equals the total load of x if that value lies in segment s. For this definition, if the total load equals $\widehat{x}$ and $\widehat{x}$ lies in segment $\widehat{s}$, $x^{\widehat{s}} = \widehat{x}$ and $x^s = 0$ for all segments $s \neq \widehat{s}$. As in the incremental formulation, $y^s = 1$ if $x^s > 0$, and $y^s = 0$ otherwise, but in this formulation at most one y^s will equal one. With this notation, the multiple choice model has the objective function $g(x) = \sum_s c^s x^s + f^s y^s$ and the constraints:

$$x = \sum_s x^s \tag{5}$$

$$b^{s-1}y^s \leq x^s \leq b^s y^s \tag{6}$$

$$\sum_s y^s \leq 1 \tag{7}$$

$$y^s \in \{0,1\}. \tag{8}$$

We refer to the system defined by (5)-(8) as M_x and its LP relaxation as $LP(M_x)$.

2.3 Convex Combination Model

The third formulation we examine is a modification of a formulation described in textbooks by Vajda [19] and Dantzig [11], and that appears as early as 1960 [10]. The original formulation was intended for continuous cost functions, so we modify it to handle arbitrary (lower semi-continuous) discontinuous functions. This formulation makes use of the fact that in a piecewise linear cost function, the cost of a load that lies in segment s is a convex combination of the cost of the two endpoints, b^{s-1} and b^s of segment s. If we define multipliers, μ^s and λ^s, as the weights on these two endpoints, then the objective function can be written as $g(x) = \sum_s \mu^s(c^s b^{s-1} + f^s) + \lambda^s(c^s b^s + f^s)$. The constraints are:

$$x = \sum_s (\mu^s b^{s-1} + \lambda^s b^s) \tag{9}$$

$$\mu^s + \lambda^s = y^s \tag{10}$$

$$\sum_s y^s \leq 1 \tag{11}$$

$$\mu^s, \lambda^s \geq 0, \quad y^s \in \{0,1\}. \tag{12}$$

In this formulation, the y variables have the same interpretation as in the multiple choice model. Constraint (11) assures that at most one of the y variables has value one. Constraint (10) assures that $\mu^s + \lambda^s = 1$ for the segment corresponding to the positive y^s variable, and that μ^s and λ^s are both zero for the other segments. Constraint (9) defines x to be the convex combination of the two endpoints defined by μ and λ. We will denote the system defined by (9)-(12) as C_x and its LP relaxation as $LP(C_x)$.

3 Comparing the Three Models

Given that all three of the previous models are valid, it is natural to ask if one is better than another. An important measure for assessing the quality of a MIP formulation is the strength of its LP relaxation. The following result characterizes the LP relaxations of these three formulations.

Proposition 1 *The LP relaxations of the incremental, multiple choice, and convex combination formulations are equivalent in the sense that they each approximate the cost function, $g(x)$, with its lower convex envelope.*

To establish this result, we need to show that for any load $\widehat{x}$, the objective value of the LP relaxation obtained by optimally choosing the other variables is given by the lower convex envelope of the cost function. We consider each formulation separately.

Proof of Proposition 1: Convex Combination Formulation

By relaxing the integrality restriction on the y variables, we can combine constraints (10) and (11) into $\sum_s(\mu^s + \lambda^s) \leq 1$ and we can eliminate the y variables. Any representation of $\widehat{x}$ as a convex combination of the breakpoints, therefore, provides a feasible solution. The cost minimizing convex combination is given by the lower convex envelope of the cost function. ■

Proof of Proposition 1: Multiple Choice Formulation

1. We prove the result by first showing that every extreme point of $LP(M_{\widehat{x}})$=$\{(x,y) : \sum_s x^s = \widehat{x}$, $b^{s-1}y^s \leq x^s \leq b^s y^s$, $\sum_s y^s \leq 1$, $y^s \geq 0\}$ is a convex combination of two endpoints of the piecewise linear segments. We first note two facts, (a) and (b).

(a) Every extreme point of the polyhedron P=$\{(x,y) : b^{s-1}y^s \leq x^s \leq b^s y^s, \sum_s y^s \leq 1, y^s \geq 0\}$ is an endpoint of one of the segments of the piecewise linear cost function. To see this result, suppose we minimize some cost function $\sum_s c^s x^s + f^s y^s$ over the polyhedron P. Note that if $c^s \geq 0$, then $x^s = b^{s-1}y^s$ in some optimal solution of the linear program and if $c^s \leq 0$, then $x^s = b^s y^s$ in some optimal solution of the linear program. Therefore, we can express each x^s in terms of the y^s

variables, and eliminate the x^s variables and the $b^{s-1}y^s \leq x^s \leq b^s y^s$ constraints from the model. The resulting problem has a linear objective function and the single constraint $\sum_s y^s \leq 1$ in the nonnegative y variables. Since the problem has a single constraint, it has a solution with at most one $y^s = 1$ and all other y variables at value zero. Since any such point with either $x^s = b^{s-1}y^s$ or $x^s = b^s y^s$ is an endpoint of a segment of $g(x)$, and we have shown that such a solution is optimal for every choice of the cost coefficients, we conclude that any extreme point of P corresponds to an endpoint of a segment. Conversely, since we can choose the cost coefficients so that any endpoint of a segment is the unique minimizing solution, any endpoint of a segment corresponds to an extreme point of P.

(b) It is easy to see that if Q is a bounded polyhedron and we let $wz = \widehat{w}$ be any linear equation, then every extreme point in the polyhedron $\widehat{Q}=\{z \in Q : wz = \widehat{w}\}$ is a convex combination of at most two extreme points of the polyhedron Q.

The results (a) and (b) imply that every extreme point of $LP(M_{\widehat{x}})$ is a convex combination of at most two endpoints of the piecewise linear segments.

2. Next we note that the convexity of P implies that if $\widehat{x}$ is a convex combination of two segment breakpoints of the function $g(x)$, then the corresponding solution of P is feasible in $LP(M_{\widehat{x}})$. That is, let $\widehat{x}$ be a convex combination of two segment breakpoints, b^{s_1} and b^{s_2}, i.e., $\widehat{x} = \lambda b^{s_1} + (1-\lambda)b^{s_2}$ for some $0 \leq \lambda \leq 1$. Let $(x,y)(s_i) = (x(s_i), y(s_i))$ be the extreme point solution of P corresponding to b^{s_i}, $i = 1,2$, that is, $x^s(s_i) = \{b^{s_i}$,if $s = s_i$, and 0,if $s \neq s_i\}$ and $y^s(s_i) = \{1$,if $s = s_i$, and 0,if $s \neq s_i\}$. Let $(x,y)= \lambda((x,y)(s_1)) + (1-\lambda)((x,y)(s_2))$. Since P is convex, $(x,y) \in P$ and, in addition, $\sum_s x^s = \lambda b^{s_1} + (1-\lambda)b^{s_2} = \widehat{x}$. Therefore, $(x,y) \in LP(M_{\widehat{x}})$.

3. Point (1) implies that if we evaluate the objective function at an extreme point of $LP(M_{\widehat{x}})$, then we are taking convex combinations of the endpoints of two segments of the cost function. When minimizing, we choose the convex combination with least possible cost. Point (2) assures that this least cost convex combination is feasible in $LP(M_{\widehat{x}})$. Therefore, the lower convex envelope of the cost function is defined by the convex combinations of the segment endpoints, and as we vary the

load $\widehat{x}$, the optimal objective value of the LP relaxation specifies the least cost convex combination of the endpoints, which defines the lower convex envelope of the cost function. ■

Because we have shown that an extreme point of $LP(M_{\widehat{x}})$ is a convex combination of two endpoints of the piecewise linear segments of $g(x)$, we have established the following result:

Corollary 2 *At an extreme point of $LP(M_{\widehat{x}})$, the set of y^s variables assumes a form where $y^s = 0$ for all s (when $\widehat{x} = 0$), or one $y^s > 0$ (when one of the endpoints is the origin), or two $y^s > 0$ and their sum is one.*

Proof of Proposition 1: Incremental Formulation

We can establish this result using the same approach as with the multiple choice formulation. We first establish an analog of (a) for this formulation; letting $\Delta^s = b^s - b^{s-1}$, we wish to show that every extreme point of the polyhedron P=$\{(x, y) : \Delta^s y^{s+1} \leq x^s \leq \Delta^s y^s,\ 0 \leq y^s \leq 1\}$ is an endpoint of one of the segments of the piecewise linear cost function. Suppose we minimize some cost function $\sum c^s x^s + f^s y^s$ over P. Note that if $c^s \geq 0$, then $x^s = \Delta^s y^{s+1}$ in some optimal solution of the linear program and if $c^s \leq 0$ then $x^s = \Delta^s y^s$. Therefore, we can express each x^s in terms of the y^s variables and we are left with the following constraints: $0 \leq y \leq 1$ and $y^s \geq y^{s+1}$. Any nonzero extreme point solution to this system is of the form y^s=1, if $s \leq \widehat{s}$ and $y^s = 0$, if $s > \widehat{s}$, for some index $\widehat{s} \neq 0$. Using these values of y to find the values for x, we see that each such extreme point of P corresponds to an endpoint of the piecewise linear segments.

The other steps of the proof are the same as those in the proof for the multiple choice formulation so we can conclude that as we vary the load $\widehat{x}$, the objective value defines the lower convex envelope of the cost function. ■

Since an extreme point of $LP(I_{\widehat{x}})$ is a convex combination of two points both having the form $(0, 0, ..., 0)$ or $(1, 1, ..., 1, 0, 0, ..., 0)$ for the y vector, this proof establishes the following result:

Corollary 3 *At an extreme point of $LP(I_{\widehat{x}})$, the set of y^s variables will be of the form $y^s = 0$ for all s (when $\widehat{x} = 0$), or $y^s = \{\overline{y}$, if $s \leq \widehat{s}$ and 0, if $s > \widehat{s}\}$ for some index $\widehat{s}$ and constant $0 \leq \overline{y} \leq 1$*

(when one of the endpoints is the origin), or $y^s = \{1,$ *if* $s \leq \widehat{t}_1$, $\overline{y}$, *if* $\widehat{t}_1 < s \leq \widehat{t}_2$ *and* 0, *if* $s > \widehat{t}_2\}$ *for some indices* $\widehat{t}_1, \widehat{t}_2$, *and for some constant* $0 \leq \overline{y} \leq 1$.

Proposition 1 establishes the equivalency of the LP relaxations of the three formulations. Another way of viewing the equivalency of these formulations is to provide a translation between feasible points of each LP relaxation.

Proposition 4 *The LP relaxations of the incremental, multiple choice, and convex combination formulations are equivalent, in the sense that we can translate a feasible solution of one LP relaxation into a feasible solution to the others with the same cost.*

Proof: We establish this result by showing how to translate any feasible solution of $LP(M_x)$ to a feasible solution of either $LP(I_x)$ or $LP(C_x)$ with the same cost, and conversely how to translate feasible solutions of either $LP(I_x)$ or $LP(C_x)$ to a feasible solution of $LP(M_x)$ with the same cost.

To prove the equivalence of the multiple choice and incremental models we will use the following fact. Suppose two formulations have bounded feasible regions and that we show that each extreme point solution of a formulation *F1* corresponds to a feasible solution (the "corresponding solution") of the formulation *F2* with the same cost (and conversely). Then since any feasible solution of the formulation *F1* is a convex combination of its extreme point solutions, the same convex combination of the corresponding solutions will provide a feasible solution with the same cost in formulation *F2*. Therefore, once we have shown that each extreme point of each formulation corresponds to a feasible solution of the other formulation with the same cost, we would have shown that the two formulations are equivalent.

Multiple Choice $\longrightarrow$ Convex Combination

Consider a feasible solution, (x, y), to $LP(M_x)$. For this solution to be feasible, $b^{s-1}y^s \leq x^s \leq b^s y^s$ which implies that for some $0 \leq \alpha^s \leq 1$, $x^s = \alpha^s b^{s-1} y^s + (1 - \alpha^s) b^s y^s$. Let $\mu^s = \alpha^s y^s$ and $\lambda^s = (1 - \alpha^s) y^s$. Then $\mu^s + \lambda^s = y^s$ and $x^s = \mu^s b^{s-1} + \lambda^s b^s$. Moreover, the constraint $x = \sum_s x^s$

implies that $x = \sum_s (\mu^s b^{s-1} + \lambda^s b^s)$. Therefore, (x, y, μ, λ) is feasible for $LP(C_x)$. The cost of this solution is $\sum_s \mu^s (c^s b^{s-1} + f^s) + \lambda^s (c^s b^s + f^s) = \sum_s c^s (\mu^s b^{s-1} \lambda^s b^s) + f^s (\mu^s + \lambda^s) = \sum_s c^s x^s + f^s y^s$, which is equal to the cost of (x, y), the solution to $LP(M_x)$.

Convex Combination $\longrightarrow$ Multiple Choice

Consider a feasible solution (x, y, μ, λ) to $LP(C_x)$. Define $x^s = \mu^s b^{s-1} + \lambda^s b^s$. The conditions $b^{s-1} \leq b^s$ and $\mu^s + \lambda^s = y^s$ imply that $b^{s-1} y^s \leq x^s \leq b^s y^s$. Therefore, (x, y) is feasible for $LP(M_x)$. As shown above, the cost of this solution is the same as the cost of (x, y, μ, λ), the solution to $LP(C_x)$.

Multiple Choice $\longrightarrow$ Incremental

To simplify the notation, we let (w, v) represent the variables in $LP(I_x)$. Consider (x, y), an extreme point solution of $LP(M_x)$. We derive a feasible solution to $LP(I_x)$ with the same cost. The translation is more complex than the previous ones because the continuous and binary variables have different interpretations in each formulation. Let $\Delta^s = b^s - b^{s-1}$. We consider two feasible solutions of $LP(I_x)$ denoted $((w, v)(\widehat{s})) = (w(\widehat{s}), v(\widehat{s}))$ and $((w, v)(\widehat{s} - 1)) = (w(\widehat{s} - 1), v(\widehat{s} - 1))$, which are defined as $w^s(\widehat{s}) = \{\Delta^s$, if $s \leq \widehat{s}$ and 0, if $s > \widehat{s}\}$, $v^s(\widehat{s}) = \{1$, if $s \leq \widehat{s}$ and 0, if $s > \widehat{s}\}$, $w^s(\widehat{s} - 1) = \{\Delta^s$, if $s \leq \widehat{s} - 1$ and 0, if $s > \widehat{s} - 1\}$, and $v^s(\widehat{s} - 1) = \{1$, if $s \leq \widehat{s}$ and 0, if $s > \widehat{s}\}$, for some index $\widehat{s} \neq 0$.

Corollary 2 characterizes extreme point solutions of $LP(M_x)$ and provides us with two distinct cases when $x > 0$ (the case $x = 0$ is trivial); either one y^s variable is positive or two are positive and their sum is one. To consider these two cases together, we modify the two formulations. We introduce a dummy segment, s^0. We let $c^0 = f^0 = \widehat{f}^0 = b^0 = b^{-1} = \Delta^0 = 0$. In the multiple choice formulation we let $y^0 = 1 - \sum y^s$ and $x^0 = 0$. Now all extreme point solutions of $LP(M_x)$ have two positive y^s variables, $y^{\widehat{s}}$ and $y^{\widehat{t}}$ with $y^{\widehat{s}} + y^{\widehat{t}} = 1$ (assume that $\widehat{s} < \widehat{t}$). Therefore, $x^{\widehat{s}} \geq 0$ and $x^{\widehat{t}} > 0$. In addition, $x^{\widehat{s}} = b^{\widehat{s}-1} y^{\widehat{s}}$ or $x^{\widehat{s}} = b^{\widehat{s}} y^{\widehat{s}}$, and $x^{\widehat{t}} = b^{\widehat{t}-1} y^{\widehat{t}}$ or $x^{\widehat{t}} = b^{\widehat{t}} y^{\widehat{t}}$.

We define (w, v), a solution to $LP(I_x)$ which now includes w^0 and v^0 to account for the dummy segment. Let $v^s = 1, \forall\, s \leq \widehat{s}$, $v^s = y^{\widehat{t}}, \forall\, \widehat{s} < s \leq \widehat{t}$, and $v^s = 0, \forall\, s > \widehat{t}$. For $s \leq \widehat{s} - 1$, let $w^s = \Delta^s$. If $x^{\widehat{s}} = b^{\widehat{s}} y^{\widehat{s}}$, let $w^{\widehat{s}} = \Delta^{\widehat{s}}$, but if $x^{\widehat{s}} = b^{\widehat{s}-1} y^{\widehat{s}}$, let $w^{\widehat{s}} = \Delta^{\widehat{s}} y^{\widehat{t}}$. For $\widehat{s} < s < \widehat{t}$, let $w^s = \Delta^s y^{\widehat{t}}$. If $x^{\widehat{t}} = b^{\widehat{t}} y^{\widehat{t}}$, let $w^{\widehat{t}} = \Delta^{\widehat{t}} y^{\widehat{t}}$, but if $x^{\widehat{t}} = b^{\widehat{t}-1} y^{\widehat{t}}$, let $w^{\widehat{t}} = 0$. Finally, for $s > \widehat{t}$, let $w^s = 0$.

We first show that (w, v) is feasible for $LP(I_x)$ by writing it as a convex combination of two feasible points (a direct algebraic approach also demonstrates feasibility). If $x^{\widehat{s}} = b^{\widehat{s}} y^{\widehat{s}}$ and $x^{\widehat{t}} = b^{\widehat{t}} y^{\widehat{t}}$, then $(w,v) = y^{\widehat{s}}\ ((w, v)(\widehat{s})) + y^{\widehat{t}}\ ((w, v)(\widehat{t}))$. If $x^{\widehat{s}} = b^{\widehat{s}-1} y^{\widehat{s}}$ and $x^{\widehat{t}} = b^{\widehat{t}} y^{\widehat{t}}$, then $(w,v) = y^{\widehat{s}}$ $((w, v)(\widehat{s} - 1)) + y^{\widehat{t}}((w, v)(\widehat{t}))$. The other two cases follow similarly.

We now show the solution $(w, v) \in LP(I_x)$ has the same cost as $(x, y) \in LP(M_x)$. We consider only the case $x^{\widehat{s}} = b^{\widehat{s}} y^{\widehat{s}}$ and $x^{\widehat{t}} = b^{\widehat{t}} y^{\widehat{t}}$, since the other cases are similar. Indeed, the cost of (w, v) is $\sum_s c^s w^s + \widehat{f}^s v^s = \sum_{s \leq \widehat{s}} (c^s \Delta^s + \widehat{f}^s) + \sum_{\widehat{s} < s \leq \widehat{t}} (c^s \Delta^s y^{\widehat{t}} + \widehat{f}^s y^{\widehat{t}}) = \sum_{s \leq \widehat{s}} (c^s \Delta^s + \widehat{f}^s) +$ $y^{\widehat{t}} \sum_{\widehat{s} < s \leq \widehat{t}} (c^s \Delta^s + \widehat{f}^s) = (f^{\widehat{s}} + c^{\widehat{s}} b^{\widehat{s}}) + y^{\widehat{t}} [(f^{\widehat{t}} + c^{\widehat{t}} b^{\widehat{t}}) - (f^{\widehat{s}} + c^{\widehat{s}} b^{\widehat{s}})] = (1 - y^{\widehat{t}})(f^{\widehat{s}} + c^{\widehat{s}} b^{\widehat{s}}) + y^{\widehat{t}} (f^{\widehat{t}} + c^{\widehat{t}} b^{\widehat{t}}) =$ $y^{\widehat{s}} (f^{\widehat{s}} + c^{\widehat{s}} b^{\widehat{s}}) + y^{\widehat{t}} (f^{\widehat{t}} + c^{\widehat{t}} b^{\widehat{t}}) = c^{\widehat{s}} b^{\widehat{s}} y^{\widehat{s}} + c^{\widehat{t}} b^{\widehat{t}} y^{\widehat{t}} + f^{\widehat{s}} y^{\widehat{s}} + f^{\widehat{t}} y^{\widehat{t}} = c^{\widehat{s}} x^{\widehat{s}} + c^{\widehat{t}} x^{\widehat{t}} + f^{\widehat{s}} y^{\widehat{s}} + f^{\widehat{t}} y^{\widehat{t}}$ which is the cost of (x, y).

Incremental $\longrightarrow$ Multiple Choice

We now consider an extreme point (w, v) of $LP(I_x)$ and derive, (x, y), a feasible solution for $LP(M_x)$ with the same cost. We consider two feasible solutions of $LP(M_x)$ denoted $((x, y)(\widehat{s})) = (x(\widehat{s}), y(\widehat{s}))$ and $((x, y)(\widehat{s} - 1)) = (x(\widehat{s} - 1), y(\widehat{s} - 1))$ defined as $x^s(\widehat{s}) = \{b^{\widehat{s}},\ s = \widehat{s}$ and $0,\ s \neq \widehat{s}\}$, $y^s(\widehat{s}) = \{1,\ s = \widehat{s}$ and $0,\ s \neq \widehat{s}\}$, $x^s(\widehat{s} - 1) = \{b^{\widehat{s}-1},\ s = \widehat{s}$ and $0,\ s \neq \widehat{s}\}$ and $y^s(\widehat{s} - 1) = \{1,\ s = \widehat{s}$ and $0,\ s \neq \widehat{s}\}$, for some $\widehat{s} \neq 0$.

Corollary 3 characterizes the extreme points of $LP(I_x)$ and provides us with two cases to consider when $x > 0$. We modify the incremental formulation with the dummy segment and let $w^0 = 0$ and $v^0 = 1$. Now the v variables of all extreme point solutions of $LP(I_x)$ have the form $v = \{1$, for $s \leq \widehat{s}$, and $\overline{v}$, for $\widehat{s} < s \leq \widehat{t}$ and 0, for $s > \widehat{t}\}$ for some $\widehat{s}, \widehat{t}$, and for some constant $0 \leq \overline{v} \leq 1$. Assume $\widehat{s} < \widehat{t}$. We know that either $w^{\widehat{s}} = \Delta^{\widehat{s}} v^{\widehat{s}}$ or $w^{\widehat{s}-1} = \Delta^{\widehat{s}-1} v^{\widehat{s}}$, and either $w^{\widehat{t}} = \Delta^{\widehat{t}} v^{\widehat{t}}$ or $w^{\widehat{t}-1} = \Delta^{\widehat{t}-1} v^{\widehat{t}}$.

To define the solution to $LP(M_x)$, let $y^{\widehat{t}} = \overline{v}$, $y^{\widehat{s}} = 1 - \overline{v}$, and $y^s = 0, \forall\, s \neq \widehat{s}, \widehat{t}$. If $w^{\widehat{s}} = \Delta^{\widehat{s}} v^{\widehat{s}}$, then let $x^{\widehat{s}} = b^{\widehat{s}}(1 - \overline{v})$. If $w^{\widehat{s}-1} = \Delta^{\widehat{s}-1} v^{\widehat{s}}$, then let $x^{\widehat{s}} = b^{\widehat{s}-1}(1 - \overline{v})$. If $w^{\widehat{t}} = \Delta^{\widehat{t}} v^{\widehat{t}}$, then let $x^{\widehat{t}} = b^{\widehat{t}} \overline{v}$. If $w^{\widehat{t}-1} = \Delta^{\widehat{t}-1} v^{\widehat{t}}$, then let $x^{\widehat{t}} = b^{\widehat{t}-1} \overline{v}$. For $s \neq \widehat{s}, \widehat{t}$, let $x^s = 0$.

We show that this solution is feasible by writing it as a convex combination of two feasible points. Specifically, if $w^{\widehat{s}} = \Delta^{\widehat{s}} v^{\widehat{s}}$ and $w^{\widehat{t}} = \Delta^{\widehat{t}} v^{\widehat{t}}$, then $(x, y) = (1 - \overline{v})((x, y)(\widehat{s})) + \overline{v}((x, y)(\widehat{t}))$. If $w^{\widehat{s}-1} = \Delta^{\widehat{s}-1} v^{\widehat{s}}$ and $w^{\widehat{t}} = \Delta^{\widehat{t}} v^{\widehat{t}}$, then $(x, y) = (1 - \overline{v})((x, y)(\widehat{s} - 1)) + \overline{v}((x, y)(\widehat{t}))$. We can define the other two cases similarly.

To show that the solution $(x, y) \in LP(M_x)$ has the same cost as $(w, v) \in LP(I_x)$, we notice that these two solutions have the same structure as they did in our translation from $LP(M_x)$ to $LP(I_x)$.

We now have a valid mapping for extreme points of the two polyhedrons $LP(M_x)$ and $LP(I_x)$. Therefore, as we have noted, we can map any feasible solution to one of the two formulations into a feasible solution to the other with the same objective cost. ■

4 Conclusion

Although the three basic MIP models we examined are well-known in the literature, to the best of our knowledge, no one has compared them. We showed that the LP relaxations of these formulations are equivalent, and that each approximates the cost function by its lower convex envelope. To further demonstrate the equivalency, we show how to translate solutions between the three LP relaxations.

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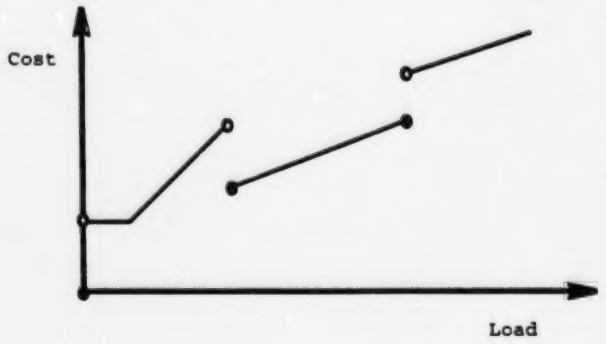

Figure 1: A Piecewise Linear Cost Function

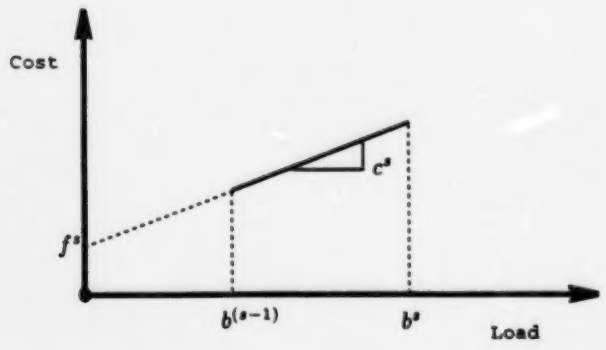

Figure 2: Notation for Each Segment